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Bennet's relation:-

We consider an infinite cylinder of conductive fluid with an axial current density $J_z = J_z(r)$ and a resulting azimuthal magnetic induction $B_\theta = B_\theta(r)$. For simplicity the current density, the magnetic field pressure etc are assumed to depend on the distance r from the cylinder's axis.

Here viscous and gravitational forces are neglected. Thus we have eqn of motion as —

$$0 = -\vec{\nabla} p + \frac{1}{c}(\vec{J}\times\vec{B}) \quad \text{——— (1)}$$

using cylindrical co-ordinates the eqn (1) can be expressed as —

$$0 = -\frac{dp}{dr} - \frac{d}{dr}\left(\frac{B_\theta^2}{8\pi}\right) - \frac{B_\theta^2}{4\pi r}$$

$$\Rightarrow \frac{dp}{dr} = -\frac{1}{8\pi r^2}\frac{d}{dr}\left(r^2 B_\theta^2\right) \quad \text{——— (2)}$$

from Maxwells eqns we have the following relation

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} \quad \text{——— (3)}$$

using cylindrical co-ordinates eqn (3) can be expressed as —

$$\frac{1}{r}\frac{d}{dr}(r\cdot B_\theta) = \frac{4\pi}{c} J_z(r)$$

$$\Rightarrow \frac{d}{dr}(r B_\theta) = \frac{4\pi r}{c} J_z(r) \quad \text{——— (4)}$$

integrating (4) we get -

$$B_\theta\left(\cancel{\;}\right) = \frac{4\pi}{cr}\int_0^r r J_z(r)\,dr$$

$$= \frac{2I}{cr} \qquad (5)$$

where $I = 2\pi\int_0^r r J_z(r)\,dr$

Integrating (2) we have the solution for pressure as -

$$p = p_0 - \frac{1}{8\pi}\int_0^{r} \frac{1}{r^2}\frac{d}{dr}\left(r^2 B_\theta^2\right)dr \qquad (6)$$

when p_0 is the pressure of the fluid at $r=0$ that is on z axis. If the matter is confined to $r \le R$, the pressure drops to zero at $r=R$. Consequently, the axial pressure p_0 is given by

$$p_0 = \frac{1}{8\pi}\int_0^{R} \frac{1}{r^2}\frac{d}{dr}\left(r^2 B_\theta^2\right)dr \qquad \begin{array}{l} r=0 \\ p=0 \end{array} \qquad (7)$$

using (7) from eqn (6) we get -

$$p = \frac{1}{8\pi}\int_r^R \frac{1}{r^2}\frac{d}{dr}\left(r^2 B_\theta^2\right)dr \qquad (8)$$

The average pressure $\langle p \rangle$ can be defined as -

$$\langle p \rangle = \frac{1}{\pi R^2}\int_0^R (2\pi r)\, p\, dr \qquad (9)$$

integrating by parts we have -

$$\langle p \rangle = \frac{1}{\pi R^2}\left[2\pi\left\{\left(\frac{p r^2}{2}\right)_0^R - \int_0^r \frac{dp}{dr}\cdot\frac{r^2}{2}\,dr\right\}\right]$$

$$= \frac{2}{R^2}\left[0 - \frac{1}{2}\int_0^R r^2 \frac{dP}{dr}\, dr\right]$$

$$= -\frac{1}{R^2}\int_0^R r^2 \frac{dP}{dr}\, dr$$

$$= -\frac{1}{R^2}\int_0^R r^2 \left(-\frac{1}{8\pi r^2}\frac{d}{dr}\left(r^2 B_\theta^2\right)\right) dr$$

(using ②)

$$= \frac{1}{8\pi R^2}\int_0^R \frac{d}{dr}\left(r^2 \cdot \frac{4I^2}{c^2 r^2}\right) dr$$

using ⑤

$$= \frac{1}{8\pi R^2}\cdot \frac{4I^2}{c^2}\int_0^R \frac{d}{dr}\left(\frac{4I^2}{c^2}\right)$$

$$= \frac{1}{8\pi R^2}\cdot \frac{4I^2}{c^2}$$

$$= \frac{I^2}{2\pi R^2 c^2} \qquad (10)$$

Note that the average pressure $\langle P \rangle$ of the matter is equal to the magnetic pressure $\frac{B^2}{8\pi}$ at the surface of the cylinder.

If the electrons and ions are at the same temperature T and number density of electron per unit length N_e, then N_e can be expressed as

$$N_e = \int_0^R (2\pi r)\, N\, dr$$

where N is the total number density and can be expressed as -

$$N = \frac{P}{2kT}$$

therefore,

$$Ne = \int_0^R (2\pi r) \cdot \frac{P}{2kT}\, dr$$

$$= \frac{\pi}{kT}\int_0^R r P\, dr$$

$$Ne = \frac{\pi}{kT}\left\{ \left(\frac{r^2}{2} P\right)_0^R - \frac{1}{2}\int_0^R \frac{dP}{dr} r^2\, dr \right\}$$

$$= \frac{-\pi}{2kT}\int_0^R r^2 \cdot \left(\frac{-1}{8\pi r^2} \frac{d}{dr}\left(r^2 B_\theta^2\right) dr\right\}$$

$$= + \frac{1}{16kT}\int_0^R \frac{d}{dr}\left(r^2 B_\theta^2\right) dr$$

$$= \frac{1}{16kT}\int_0^R \frac{d}{dr}\left(r^2 \frac{4I^2}{c^2 r^2}\right) dr$$

$$= \frac{1}{16kT} \frac{4I^2}{c^2}$$

$$Ne = \frac{I^2}{4kTc^2}$$

$$\Rightarrow I^2 = 4kTc^2 Ne \qquad (11)$$

This is the Bennet's relation for linear pinch. It shows that the kinetic temperature T is proportional to the square of the total current I flowing in the discharge tube and inversely to the line density N_e to the particle.

$$T = \left(\frac{1}{4ke^2}\right)\frac{T^2}{N_e}$$

Defⁿ (Note)

Derivation of Generalised OHM'S LAW

Let us define the mean velo v and mass ~~velocity~~ density ρ_m, current density J and charge density ρ_q

$$v = \frac{m_i n_i v_i + m_e n_e v_e}{m_i n_i + m_e n_e} \quad \text{—— (1)}$$

$$J = e(n_i v_i - n_e v_e) \quad \text{—— (2)}$$

$$\rho_m = m_i n_i + m_e n_e \quad \text{—— (3)}$$

$$\rho_q = e(n_i - n_e) \quad \text{—— (4)}$$

Now the momentum transfer eqns for electrons and ions are

$$\frac{\partial \vec{v}_e}{\partial t} + (\vec{v}_e \cdot \vec{\nabla})\vec{v}_e = -\frac{\vec{\nabla} P_e}{m_e n_e} - \frac{e}{m_e}\left[\vec{E} + \frac{1}{c}(\vec{v}_e \times \vec{B}_0)\right] \quad (11)$$

+ Collision term.

$$\frac{\partial \vec{v}_i}{\partial t} + (\vec{v}_i \cdot \vec{\nabla})\vec{v}_e = -\frac{\vec{\nabla} p_i}{m_i n_i} + \frac{e}{m_i}\left(\vec{E} + \frac{1}{c}(\vec{v}_i \times \vec{B}_0)\right) \quad \text{—— (5)}$$

+ Collision term.

Multiplying equation – (5) by $-en_e$ and (6) by en_i and then adding we get –

$$-en_e \frac{\partial v_e}{\partial t} + en_i \frac{\partial v_i}{\partial t} - en_e (v_e \cdot \nabla) v_e$$

$$+ en_i (v_i \cdot \nabla) v_i = e\left(\frac{\nabla P_e}{m_e} - \frac{\nabla P_i}{m_i}\right)$$

$$+ \left(\frac{e^2 n_e}{m_e} + \frac{e^2 n_i}{m_i}\right) E + \frac{1}{c}\left(\frac{e^2 n_e v_e}{m_e} + \frac{e^2 n_i v_i}{m_i}\right) \times B_0$$

+ collision term

$\longrightarrow$ (7)

We have the equation of continuity for electrons & ions as

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e v_e) = 0 \quad \text{——— (8)}$$

$$\frac{\partial n_i}{\partial t} + \nabla (n_i v_i) = 0 \quad \text{——— (9)}$$

Now

$$en_i \frac{\partial v_i}{\partial t} - en_e \frac{\partial v_e}{\partial t} = \frac{\partial}{\partial t}(en_i v_i - en_e v_e)$$

$$- v_i \frac{\partial}{\partial t}(en_i) + v_e \frac{\partial}{\partial t}(en_e)$$

$$= \frac{\partial J}{\partial t} + e v_i \nabla (n_i v_i) - e v_e \nabla (n_e v_e)$$

(using 2)

Now

$$\frac{e^2 n_e}{m_e} v_e = \frac{e}{m_e}(en_e v_e - en_i v_i) + \frac{e^2 n_i v_i}{m_e}$$

$$= \frac{e^2 n_i v_i}{m_e} - \frac{e}{m_e} j \quad \text{——— (10)}$$

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